Name: ______________________________

Instructor: __________________________

Math 10560, Practice Exam 3
April 24, 2023

- The Honor Code is in effect for this examination. All work is to be your own.
- No calculators.
- The exam lasts for 1 hour and 15 min.
- Be sure that your name is on every page in case pages become detached.
- Be sure that you have all 13 pages of the test.

PLEASE MARK YOUR ANSWERS WITH AN X, not a circle!

1.	(a)	(b)	(c)	(d)	(e)
2.	(a)	(b)	(c)	(d)	(e)
3.	(a)	(b)	(c)	(d)	(e)
4.	(a)	(b)	(c)	(d)	(e)
5.	(a)	(b)	(c)	(d)	(e)
6.	(a)	(b)	(c)	(d)	(e)
7.	(a)	(b)	(c)	(d)	(e)
8.	(a)	(b)	(c)	(d)	(e)
9.	(a)	(b)	(c)	(d)	(e)
10.	(a)	(b)	(c)	(d)	(e)

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Multiple Choice	______
11.	______
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Total	______

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Multiple Choice

1.(7 pts.) The series

$$\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$$

This series converges conditionally. It's an alternating series with $b_n = 1/\sqrt{n}$. We have
(i) The sequence $\{b_n\}_{n=2}^{\infty}$ is decreasing since $\sqrt{n+1} > \sqrt{n}$ and thus $b_{n+1} = 1/\sqrt{n+1} < 1/\sqrt{n} = b_n$ for all $n \geq 2$.
(ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} 1/\sqrt{n} = 0$.
Thus the series converges by the Alternating Series Test. But the series $\displaystyle\sum_{n=2}^{\infty} |\frac{(-1)^{n+1}}{\sqrt{n}}| = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ diverges since it's a p series and $p = \dfrac{1}{2} < 1$.

(a) converges absolutely.

(b) diverges because the terms alternate.

(c) diverges even though $\displaystyle\lim_{n\to\infty} \frac{(-1)^{n+1}}{\sqrt{n}} = 0$.

(d) diverges because $\displaystyle\lim_{n\to\infty} \frac{(-1)^{n+1}}{\sqrt{n}} \neq 0$.

(e) does not converge absolutely but does converge conditionally.

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2.(7 pts.) Use Comparison Tests to determine which **one** of the following series is divergent.

$\sum_{n=1}^{\infty} \frac{1}{n^{(3/2)+1}}$ converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$, a p-series with $p = \frac{3}{2} > 1$.

$\sum_{n=1}^{\infty} \frac{1}{n^2+8}$ converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, a p-series with $p = 2 > 1$.

$\sum_{n=1}^{\infty} \frac{n^2-1}{n^3+100}$ diverges by limit comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$, a p-series with $p = 1$.

$\sum_{n=1}^{\infty} 7\left(\frac{5}{6}\right)^n$ converges since it is a geometric series with $|r| = \frac{5}{6} < 1$.

$\sum_{n=1}^{\infty} \frac{n}{n+1}\left(\frac{1}{2}\right)^n$ converges by comparison with $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$, a geometric series with $|r| = \frac{1}{2} < 1$.

(a) $\sum_{n=1}^{\infty} \frac{n}{n+1}\left(\frac{1}{2}\right)^n$ (b) $\sum_{n=1}^{\infty} \frac{n^2-1}{n^3+100}$ (c) $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}+1}$

(d) $\sum_{n=1}^{\infty} 7\left(\frac{5}{6}\right)^n$ (e) $\sum_{n=1}^{\infty} \frac{1}{n^2+8}$

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3.(7 pts.) Consider the following series

$$(I)\quad \sum_{n=1}^{\infty}\left(\frac{2n^2+7}{n^2+1}\right)^n \qquad (II)\quad \sum_{n=2}^{\infty}\frac{2^{1/n}}{n-1} \qquad (III)\quad \sum_{n=1}^{\infty}\frac{n!}{e^n}$$

For (I), we apply the n th root test. $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\dfrac{2n^2+7}{n^2+1}$ $= \lim_{n\to\infty}\dfrac{2+7/n^2}{1+1/n^2} = 2 > 1$. Therefore the series diverges.

$\sum_{n=2}^{\infty}\dfrac{2^{1/n}}{n-1}$ diverges by direct comparison with the series $\sum\dfrac{1}{n}$, since $\dfrac{2^{1/n}}{n-1} > \dfrac{1}{n-1} > \dfrac{1}{n}$ for all n.

For III, we apply the ratio test, $\lim_{n\to\infty}\dfrac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\dfrac{(n+1)!}{e^{n+1}}\Big/\dfrac{n!}{e^n}$ $= \lim_{n\to\infty}\dfrac{n+1}{e} = \infty > 1$. Therefore the series diverges.

Therefore they all diverge.

Which of the following statements is true?

(a) (I) converges, (II) diverges, and (III) converges.

(b) They all converge.

(c) They all diverge.

(d) (I) diverges, (II) diverges, and (III) converges.

(e) (I) converges, (II) diverges, and (III) diverges.

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4.(7 pts.) Which series below conditionally converges?

Recall that a series is conditionally convergent if it is convergent but not absolutely convergent. Note immediately that c) and e) are divergent as their terms tend not to zero as n goes to infinity. Now, b), a), and d) are convergent by the alternating series test. Further, considering the corresponding series given by taking the absolute value term wise we see that a) and b) are absolutely convergent , while d) is not. Hence d) alone is conditionally convergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n^3}}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}e^n}{\sqrt{n}}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$

(e) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}$

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5.(7 pts.) Which series below is the MacLaurin series (Taylor series centered at 0) for $\dfrac{x^2}{1+x}$?

$$\frac{x^2}{1+x} = \frac{x^2}{1-(-x)} = x^2\sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^{n+2},$$

for $|x| < 1$.

(a) $\displaystyle\sum_{n=0}^{\infty}\frac{x^{n+2}}{n+2}$ (b) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\ x^{n+2}$ (c) $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n x^{2n-2}}{n!}$

(d) $\displaystyle\sum_{n=0}^{\infty}x^{2n+2}$ (e) $\displaystyle\sum_{n=0}^{\infty}(-1)^n x^{2n}$

6.(7 pts.) Which series below is a power series for $\cos(\sqrt{x})$?

Solution: Since $\cos x = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$, we have

$$\cos(\sqrt{x}) = \sum_{n=0}^{\infty}(-1)^n\frac{(\sqrt{x})^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{(2n)!} = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \cdots .$$

(a) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{(2n)!}$ (b) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n^2+1}$ (c) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n-\frac{1}{2}}}{(2n)!}$

(d) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{(2n+1)!}$ (e) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n \sqrt{x}^n}{(2n)!}$

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7.(7 pts.) Calculate

$$\lim_{x\to 0} \frac{\sin(x^3) - x^3}{x^9}.$$

Hint: Without MacLaurin series this may be a long problem.

Since $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$, we have

$$\sin(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{(2n+1)!} = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots ,$$

and

$$\lim_{x\to 0} \frac{\sin(x^3) - x^3}{x^9} = \lim_{x\to 0} \frac{(x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots) - x^3}{x^9} = \lim_{x\to 0} \frac{-\frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots}{x^9} = -\frac{1}{6}.$$

(a) 0 (b) $\frac{9}{7}$ (c) $\frac{7}{9}$ (d) ∞ (e) $-\frac{1}{6}$

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8.(7 pts.) Find a power series representation for the the function $f(x) = \ln(1-x^2)$.
Hint: $\dfrac{d}{dx}\ln(1-x^2) = \dfrac{-2x}{1-x^2}$.

Solution:

Using the hint as a starting place, we can find the expansion for the derivative and then integrate term by term to arrive at a power series for the initial function. From our knowledge of the geometric series, we can write

$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n}, \text{ so locally } \frac{d}{dx}\ln(1-x^2) = \sum_{n=0}^{\infty} -2x^{2n+1}.$$

Then we integrate and solve for our constant of integration, $f(0) = \ln(1) = 0$, so in the end we find our power series about 0 is

$$\ln(1-x^2) = \sum_{n=0}^{\infty} \frac{(-2)x^{2n+2}}{2n+2}.$$

(a) $\displaystyle\sum_{n=0}^{\infty}(-2)(2n+1)x^{2n}$

(b) $\displaystyle\sum_{n=0}^{\infty}\frac{(-2)^n x^{2n+2}}{2n+2}$

(c) $\displaystyle\sum_{n=0}^{\infty}\frac{(-2)x^{2n+2}}{2n+2}$

(d) $\displaystyle\sum_{n=0}^{\infty}(-2)^n x^{2n}$

(e) $\displaystyle\sum_{n=0}^{\infty}\frac{(-2)^n x^{2n+1}}{2n+1}$

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9.(7 pts.) What is the fourth Taylor polynomial, $T_4(x)$, for $\cos(2x)$ with center $a = \pi$?
Solution:

$$\cos(2\pi) = 1$$
$$\cos(2x)'|_{x=\pi} = -2\sin(2\pi) = 0$$
$$\cos(2x)^{(2)}|_{x=\pi} = -4\cos(2\pi) = -4$$
$$\cos(2x)^{(3)}|_{x=\pi} = 8\sin(2\pi) = 0$$
$$\cos(2x)^{(4)}|_{x=\pi} = 16\cos(2\pi) = 16$$

Hence the Taylor polynomial at $x = \pi$ is

$$1 - 2(x-\pi)^2 + \frac{2}{3}(x-\pi)^4$$

$$\boxed{1 - 2(x-\pi)^2 + \frac{2}{3}(x-\pi)^4}$$

(a) $1 - 4(x-\pi)^2 + 16(x-\pi)^4$

(b) $1 - \frac{1}{2!}(x-\pi)^2 + \frac{1}{4!}(x-\pi)^4$

(c) $1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4$

(d) $1 - 2(x-\pi)^2 + \frac{2}{3}(x-\pi)^4$

(e) $1 + 4(x-\pi)^2 + 16(x-\pi)^4$

10.(7 pts.)

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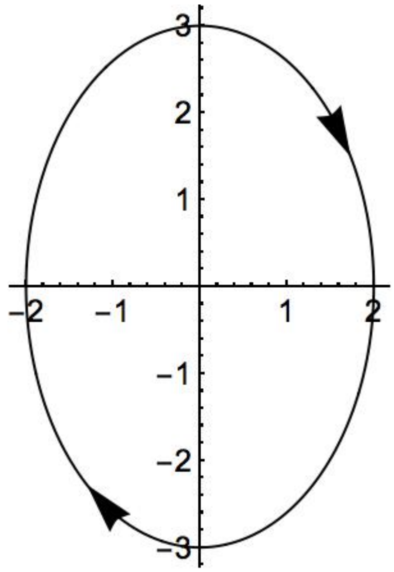

The graph of the parametric curve shown above is the graph of which of the following parametric equations?

Solution: Since the graph passes the points $(-2,0),(2,0),(0,-3),(0,3)$, only $x(t) = 2\sin(t),\quad y(t) = 3\cos(t),\quad 0 \le t \le 2\pi$ and $x(t) = 2\cos(t),\quad y(t) = 3\sin(t),\quad 0 \le t \le 2\pi$ are possible. Observe that the curve is clockwise, so $x(t) = 2\sin(t),\quad y(t) = 3\cos(t),\quad 0 \le t \le 2\pi$ is the correct one.

$$\boxed{x(t) = 2\sin(t),\ y(t) = 3\cos(t)}$$

(a) $x(t) = 3\cos(t),\quad y(t) = 2\sin(t),\quad 0 \le t \le 2\pi.$

(b) $x(t) = 2\cos(t),\quad y(t) = 3\sin(t),\quad 0 \le t \le 2\pi.$

(c) $x(t) = 2\sin(t),\quad y(t) = 3\cos(t),\quad 0 \le t \le 2\pi.$

(d) $x(t) = \dfrac{3}{2}\sin(t),\quad y(t) = \cos(t),\quad 0 \le t \le 2\pi.$

(e) $x(t) = 3\sin(t),\quad y(t) = 2\cos(t),\quad 0 \le t \le 2\pi.$

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Partial Credit

You must show your work on the partial credit problems to receive credit!

11.(11 pts.) Does the series

$$\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{2n}}$$

converge or diverge? Show your reasoning and state clearly any theorems or tests you are using.

Remark: The correct answer with no justification is worth 2 points.

Let $a_n = \frac{(n!)^n}{n^{2n}} = (\frac{n!}{n^2})^n$. Since

$$\lim_{n\to\infty} \frac{n!}{n^2} = \lim_{n\to\infty} \frac{n-1}{n} \cdot (n-2)! = \infty,$$

we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{n!}{n^2}\right)^n = \infty.$$

Hence $\lim_{n\to\infty} a_n \neq 0$. By the Test for Divergence, the series is divergent.

Another possibility is to use the Root Test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{n!}{n^2} = \infty.$$

Since the limit is > 1, the series diverges.

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12.(11 pts.) Find the radius of convergence and interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} (x-3)^n$$

Remark: The correct answer with no justification is worth 2 points.

Set $a_n = \dfrac{(-1)^n}{\sqrt{n}}(x-3)^n$. Using the Ratio Text,

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}}|x-3| = |x-3|.$$

Hence, the radius of convergence is 1, and the series converges absolutely for $|x-3| < 1$, or $2 < x < 4$.

For the end points,

when $x = 2$, the series is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which is divergent since it is a p-series with $p = \dfrac{1}{2} < 1$;

when $x = 4$, the series is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(1)^n}{\sqrt{n}}$ which is convergent since it's an alternating series, and $b_n = \dfrac{1}{\sqrt{n}}$ is decreasing and $\displaystyle\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

Hence, the interval of convergence is $2 < x \leq 4$.

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13.(11 pts.)

(a) Show that

$$\sum_{n=0}^{\infty}(-1)^n x^{2n} = \frac{1}{1+x^2}$$

provided that $|x| < 1$.

(b) Find

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)(\sqrt{3})^{2n+1}}.$$

(**Hint:** First use term-by-term integration on the series in part (a).)

(a) Since $|x| < 1$, we have $|x^2| < 1$. Hence

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}.$$

(b) Integrate both the left and right hands of (a) to get

$$\int \sum_{n=0}^{\infty}(-1)^n x^{2n} dx = \int \frac{1}{1+x^2} dx$$

$$\Rightarrow \sum_{n=0}^{\infty}\int (-1)^n x^{2n} dx = \int \frac{1}{1+x^2} dx$$

$$\Rightarrow \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} = \arctan x + C.$$

Letting $x = 0$, we have $C = 0$. Hence, we have

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} = \arctan x.$$

Let $x = \frac{1}{\sqrt{3}}$. We get

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)(\sqrt{3})^{2n+1}} = \arctan\frac{1}{\sqrt{3}} = \frac{\pi}{6}.$$

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7.	(a)	(b)	(c)	(d)	(•)
8.	(a)	(b)	(•)	(d)	(e)
9.	(a)	(b)	(c)	(•)	(e)
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Please do NOT write in this box.

Multiple Choice	________
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13.	________
Total	________